

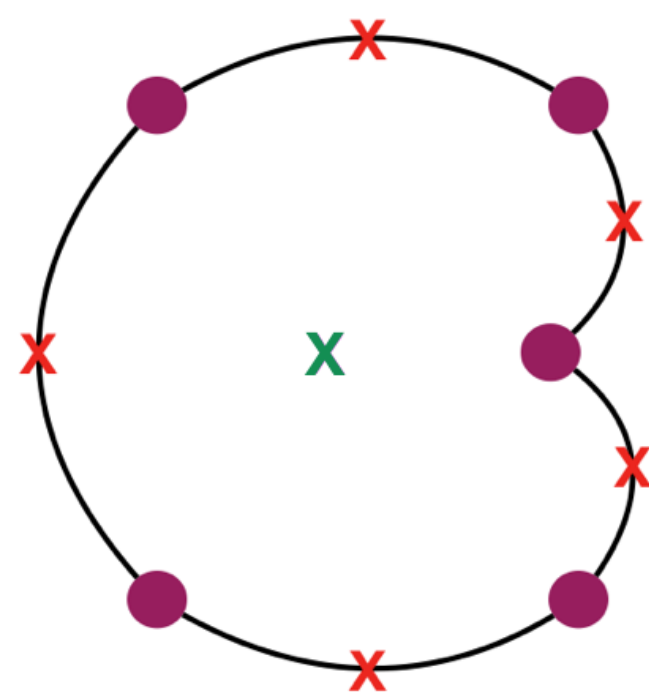
Goal

To compute the following topological information for a smooth, orientable, real surface defined by a system of polynomial equations:

1. Euler characteristic,
2. genus,
3. Betti numbers,
4. generators of the fundamental group.

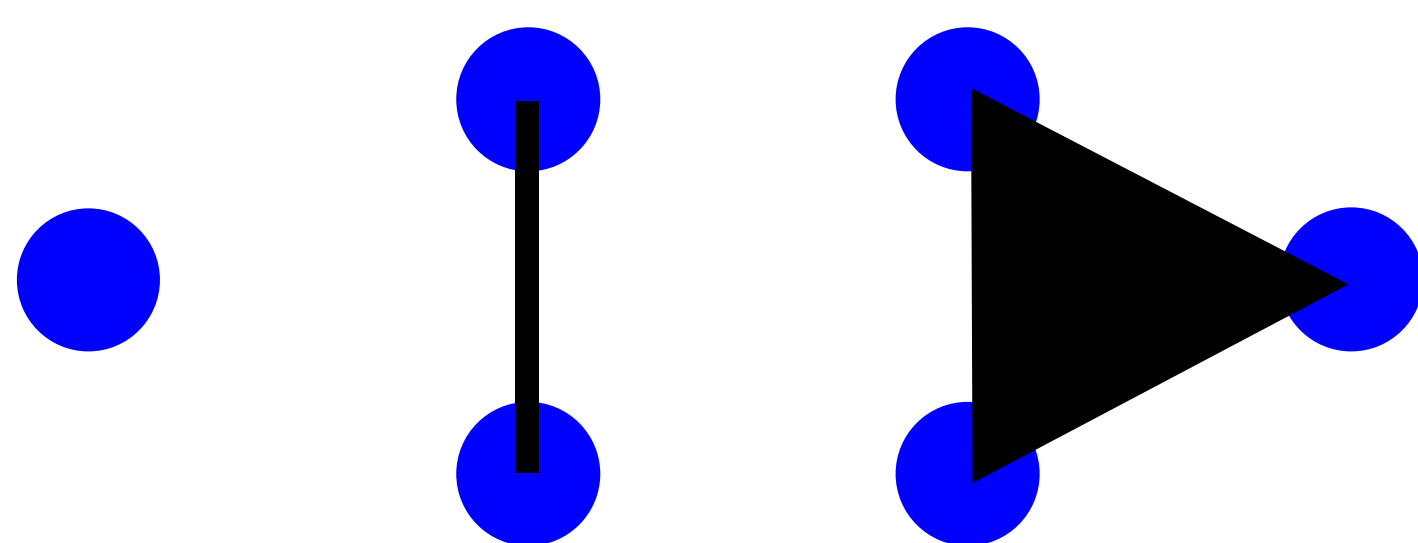
This is achieved by leveraging numerical algebraic geometry computations from `Bertini`¹ and `Bertini_real`² with topological computations in `javaPlex`⁴.

Cell Decomposition



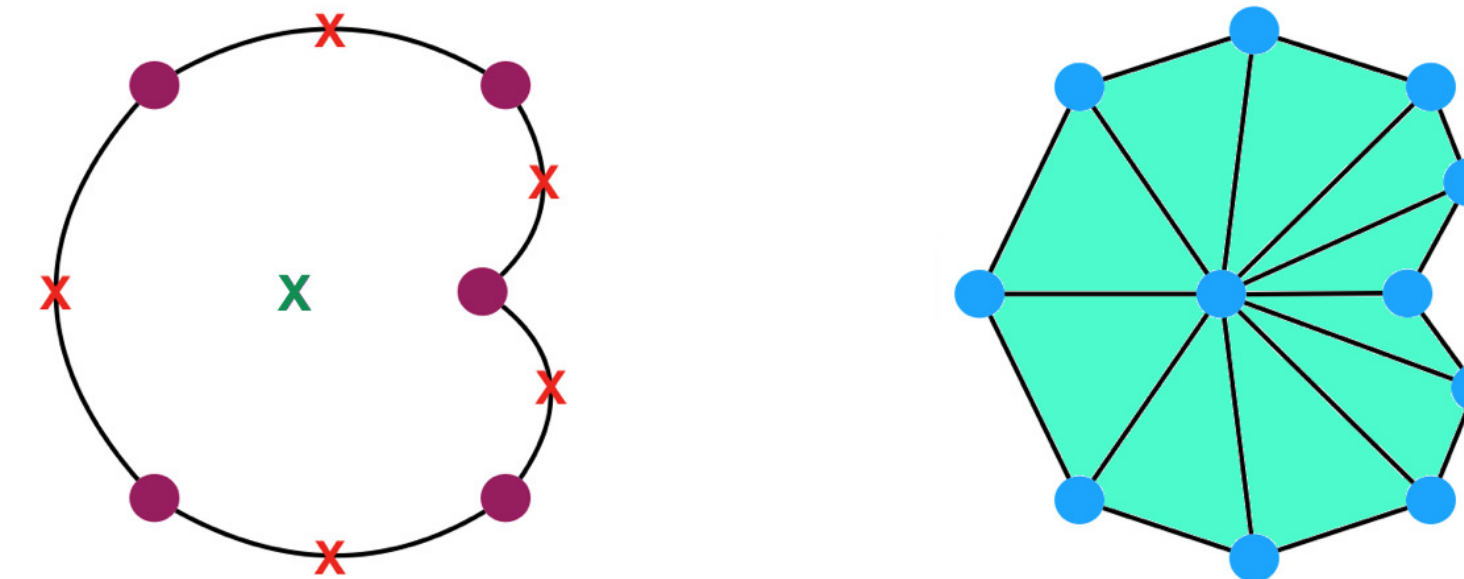
A cell decomposition is made up of a union of cells. A 2-cell is a surface patch with a generic interior point (green x) and boundary curves. Each boundary curve has a generic interior point (red x) and endpoints (maroon dots).

Simplicial Complex



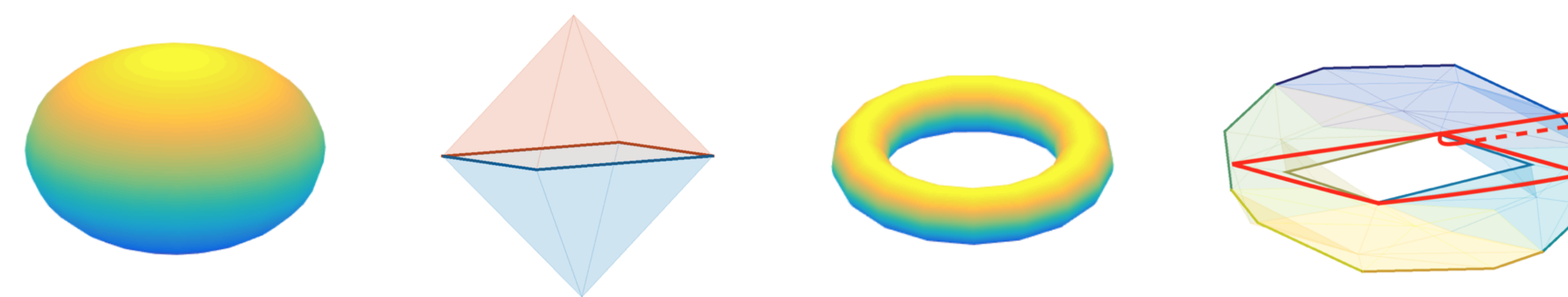
- A simplicial complex $S(X)$ is made up of vertices V , edges E , and faces F .
- Euler characteristic $\chi = V - E + F$
- Test if X is connected by computing the transitive closure $A^+ = A + A^2 + A^3 + \dots + A^V$ using Boolean arithmetic, where A is the adjacency matrix. X is connected if every entry of A^+ is 1.
- If X is connected, genus g satisfies $2 - 2g = \chi$ and the Betti numbers are $\beta_0 = \beta_2 = 1$ and $\beta_1 = 2g$. A basis of the fundamental group $\pi_1(X)$ consists of $2g$ loops.

Software



1. Run `Bertini`¹ and `Bertini_real`² to gather cell decomposition data.
2. Translate the cell decomposition data to a simplicial complex structure and compute the Euler characteristic, genus, and Betti numbers.
3. `javaPlex`⁴ computes generators of the fundamental group and confirms Betti numbers.

Sphere and Torus



- The unit sphere in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^2 = 1$:
 - The fundamental group is trivial: genus 0 with Euler characteristic 2 and Betti numbers $\beta_0 = \beta_2 = 1$ with $\beta_1 = 0$
 - Simplicial complex structure had $V = 6$ vertices, $E = 12$ edges, and $F = 8$ faces
- Torus in $\mathbb{R}^3$ defined by $(x^2 + y^2 + z^2 + 15/4)^2 - 16(x^2 + y^2) = 0$:
 - Genus 1 with Euler characteristic 0 and Betti numbers $\beta_0 = \beta_2 = 1$ with $\beta_1 = 2$
 - Simplicial complex structure had $V = 32$ vertices, $E = 96$ edges, and $F = 64$ faces
 - There are two generating loops computed by `javaPlex`:

Dimension : 1

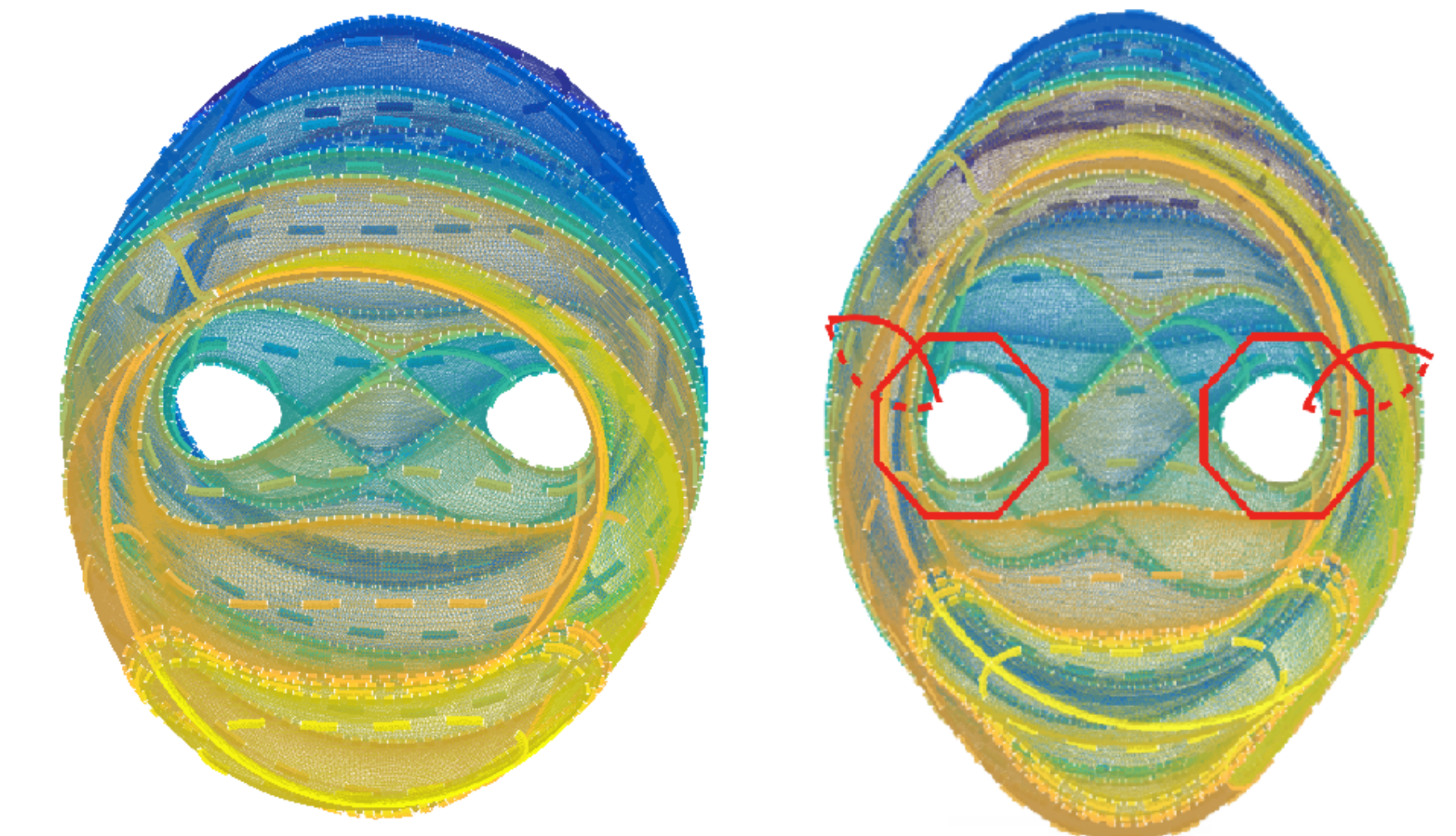
[0.0, infinity) : [1, 14] + [2, 9] + [2, 14] + [1, 9]

[0.0, infinity) : [3, 18] + [3, 17] + [1, 17] + [1, 18]

References

1. D. Bates, J. Hauenstein, A. Sommese, and C. Wampler. `Bertini`: Software for numerical algebraic geometry. Available at bertini.nd.edu.
2. D. Brake, D. Bates, W. Hao, J. Hauenstein, A. Sommese, and C. Wampler (2017). Algorithm 976: `Bertini_real`: Numerical decomposition of real algebraic curves and surfaces. *ACM Trans. Math. Softw.*, 44(1), 10. Available at bertinireal.com.
3. M. do Carmo (1976). *Differential Geometry of Curves and Surfaces*. Prentice Hall.
4. A. Tausz, M. Vejdemo-Johansson, and H. Adams (2014). `javaPlex`: A research software package for persistent (co)homology. In *Mathematical Software – ICMS 2014*, volume 8592 of *LNCIS*, pp. 129–136. Available at appliedtopology.github.io/javaplex/.

Crixxi

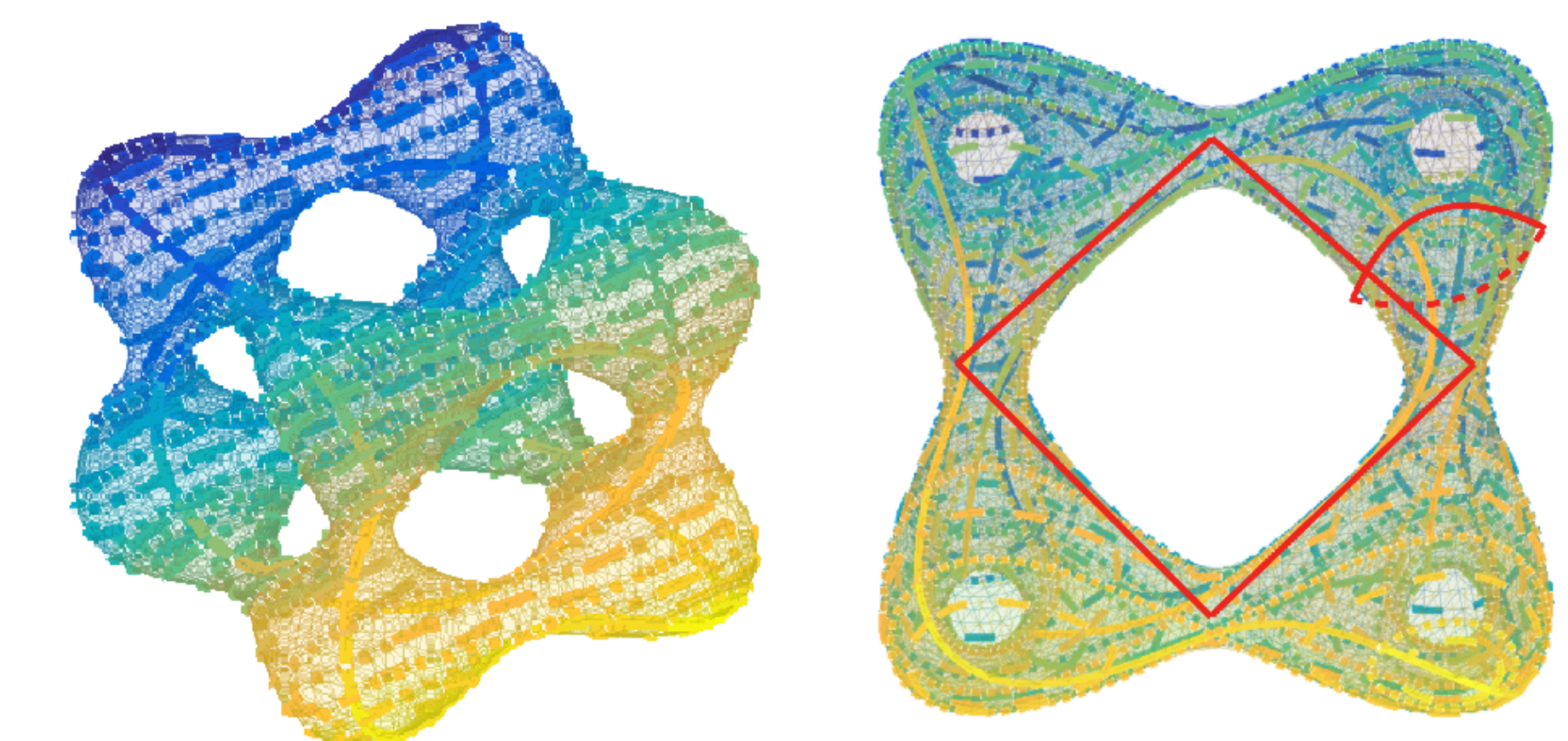


Degree 6 surface in $\mathbb{R}^3$ defined by

$$\left(\frac{1}{25}x^2 + \frac{1}{25}y^2 - 1\right)^3 + \left(\frac{1}{25}y^2 + \frac{1}{25}z^2 - 1\right)^2 = 0.$$

- Perturb the right-hand side (i.e., replace 0 with 1/10) to remove singularities yielding a real surface that is smooth and orientable³
- Genus 2 with Euler characteristic -2 and Betti numbers $\beta_0 = \beta_2 = 1$ with $\beta_1 = 4$
- Simplicial complex: $V = 346$ vertices, $E = 1044$ edges, and $F = 696$ faces
- There are 4 generating loops: 2 loops consist of 8 edges, while the others have 32 and 40 edges

Tanglecube



Degree 4 surface in $\mathbb{R}^3$ defined by

$$x^4 - 5x^2 + y^4 - 5y^2 + z^4 - 5z^2 + 11.8 = 0.$$

- Genus 5 with Euler characteristic -8 and Betti numbers $\beta_0 = \beta_2 = 1$ with $\beta_1 = 10$
- Simplicial complex: $V = 296$ vertices, $E = 912$ edges, and $F = 608$ faces
- There are 10 generating loops: 5 loops consist of 4 edges, while the others have more than 20 edges