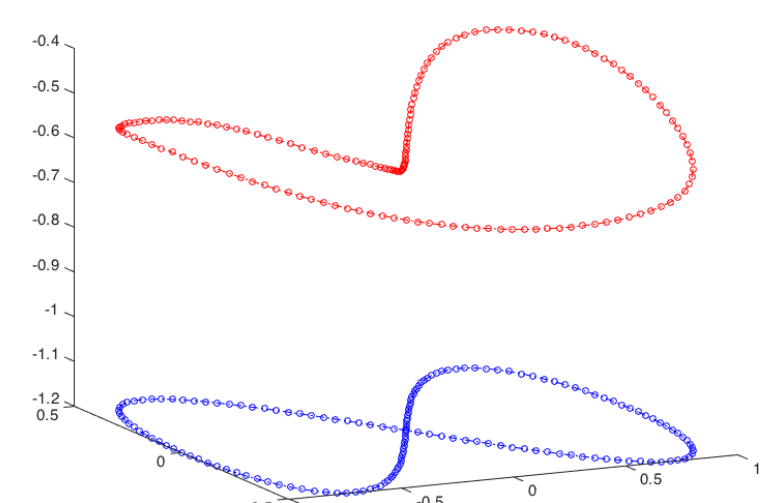


Overview

Goal: To solve PDEs on algebraic curves and surfaces, specifically handling those with singularities.

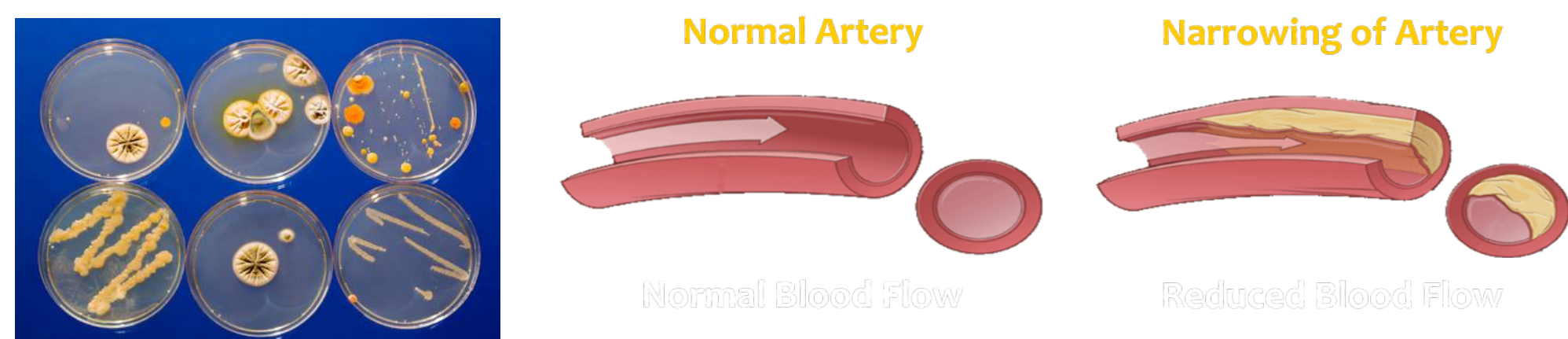
Main question: How do we choose our neighbors?



Our proposed method:

- Discretizes the curve or surface
 - Can use a witness set or a global parametrization^{1,2}
- Locally parametrizes:
 - Smooth regions - use local tangential parametrization
 - Singular regions - use method based on local geometry
- Solves PDE using finite difference method

Applications



With new technology, biomedical data is increasingly available, and can be integrated into the mathematical models used for biological processes^{2,3}. For example, in bacterial growth within petri dishes or the pressure on arterial walls.

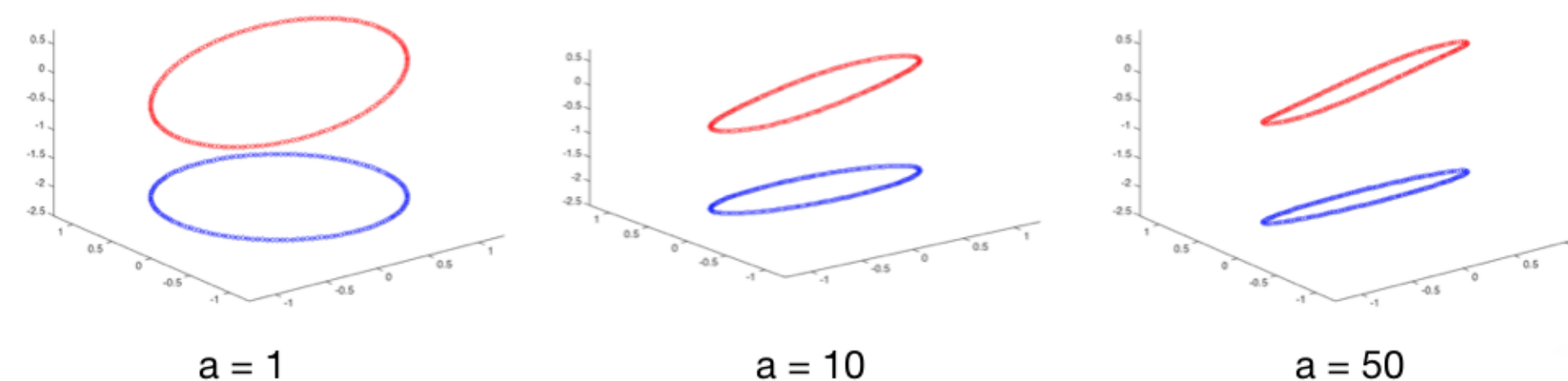
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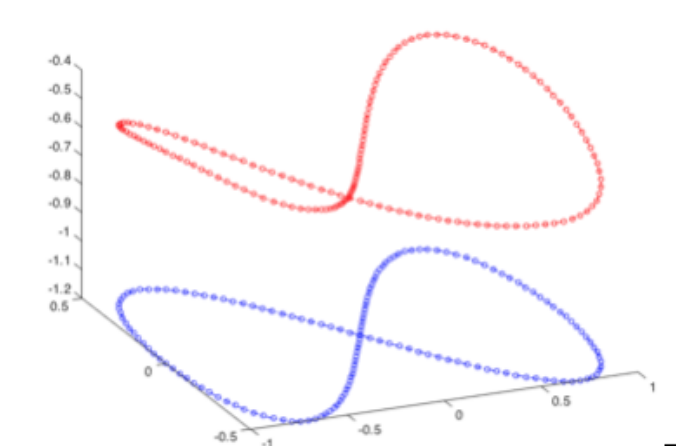
Curves

Ellipse

$$\begin{array}{l} \text{Solve } -\Delta u + u = x \\ \text{on } x^2 + ay^2 = 1 \end{array}$$



	# Grid Points	L_∞ Error	Order
a = 1	160	$9.639 \cdot 10^{-5}$	—
	320	$2.410 \cdot 10^{-5}$	2.000
	640	$6.024 \cdot 10^{-6}$	2.000
	1280	$1.506 \cdot 10^{-6}$	2.000
a = 10	160	$5.744 \cdot 10^{-4}$	—
	320	$1.442 \cdot 10^{-4}$	1.995
	640	$3.607 \cdot 10^{-5}$	1.999
	1280	$9.020 \cdot 10^{-6}$	2.000
a = 50	160	$2.459 \cdot 10^{-3}$	—
	320	$6.401 \cdot 10^{-4}$	1.942
	640	$1.630 \cdot 10^{-4}$	1.975
	1280	$4.094 \cdot 10^{-5}$	1.992



Lemniscate

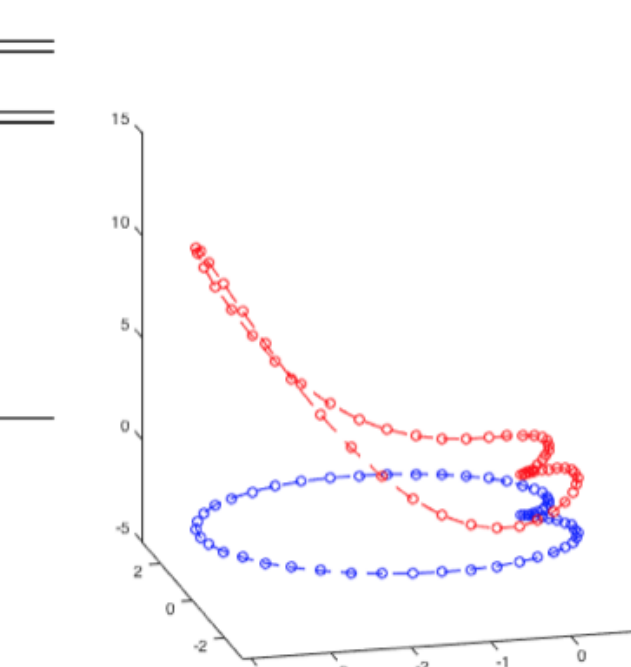
$$\begin{array}{l} \text{Solve } -\Delta u + u = f(x, y) \\ \text{on } x^4 - x^2 + y^2 = 0 \\ \text{with } f(x, y) = x^2 + xy - 1 \end{array}$$

# Grid Points	L_∞ Error	Order
160	$4.432 \cdot 10^{-4}$	—
320	$1.094 \cdot 10^{-4}$	2.019
640	$2.712 \cdot 10^{-5}$	2.012
1280	$6.735 \cdot 10^{-6}$	2.010

# Grid Points	L_∞ Error	Order
60	$2.826 \cdot 10^{-5}$	—
80	$2.348 \cdot 10^{-6}$	8.648
100	$3.656 \cdot 10^{-7}$	8.334
120	$8.196 \cdot 10^{-8}$	8.202
140	$2.339 \cdot 10^{-8}$	8.134

Cardioid

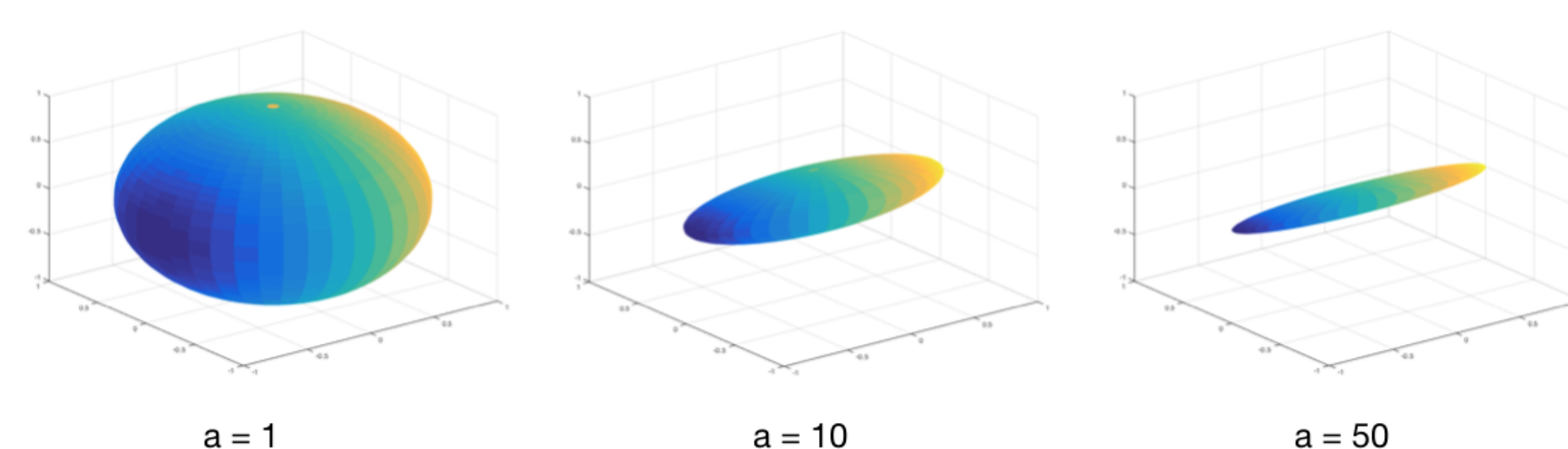
$$\begin{array}{l} \text{Solve } -\Delta u + u = f(x, y) \\ \text{on } (x^2 + y^2)^2 + 4x(x^2 + y^2) - 4y^2 = 0 \\ \text{with } f(x, y) = \frac{3607x^2 - 224xy^2 + 7662x - 53y^2 - 973}{196x^2 + 616x + 196y^2 + 1112} \end{array}$$



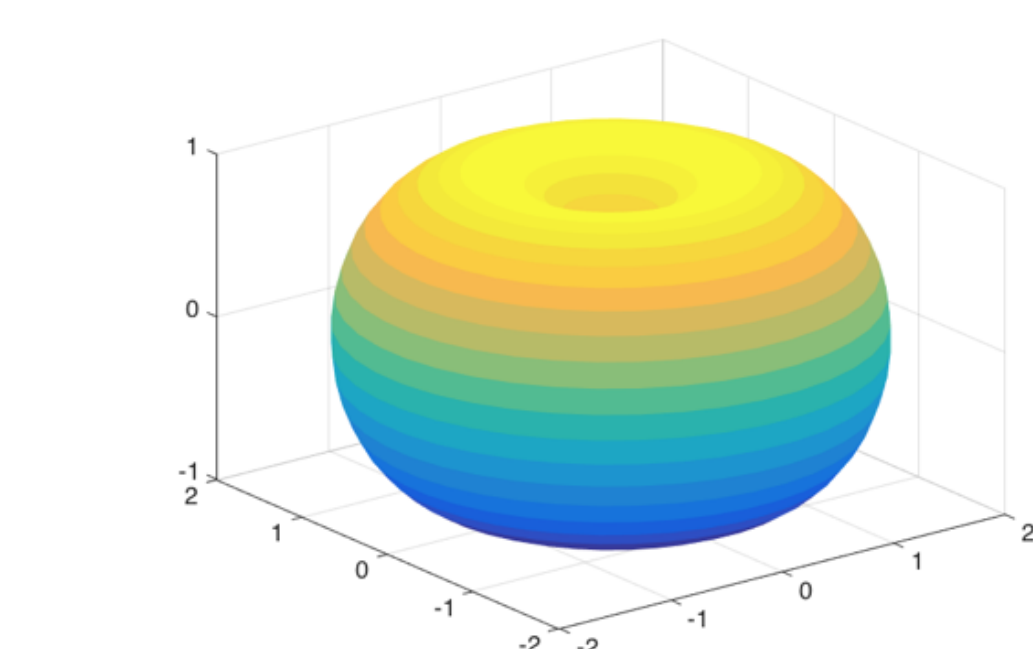
Surfaces

Ellipsoid

$$\begin{array}{l} \text{Solve } -\Delta u + u = f(x) \\ \text{on } x^2 + ay^2 + az^2 = 1 \\ \text{with } f(x) = \frac{1}{3(a+x^2(1-a))^2} [ax(2a+1) + x^3(1-a)(3a+x^2(1-a))] \end{array}$$



	# Grid Points	L_∞ Error	Order
a = 1	400	$3.462 \cdot 10^{-2}$	—
	1600	$8.655 \cdot 10^{-4}$	2.000
	6400	$2.164 \cdot 10^{-4}$	2.000
	25600	$5.410 \cdot 10^{-5}$	2.000
a = 10	400	$2.337 \cdot 10^{-2}$	—
	1600	$5.756 \cdot 10^{-3}$	2.022
	6400	$1.435 \cdot 10^{-3}$	2.004
	25600	$4.056 \cdot 10^{-4}$	1.823
a = 50	400	$3.219 \cdot 10^{-2}$	—
	1600	$1.326 \cdot 10^{-2}$	1.279
	6400	$3.606 \cdot 10^{-3}$	1.879
	25600	$9.129 \cdot 10^{-4}$	1.982



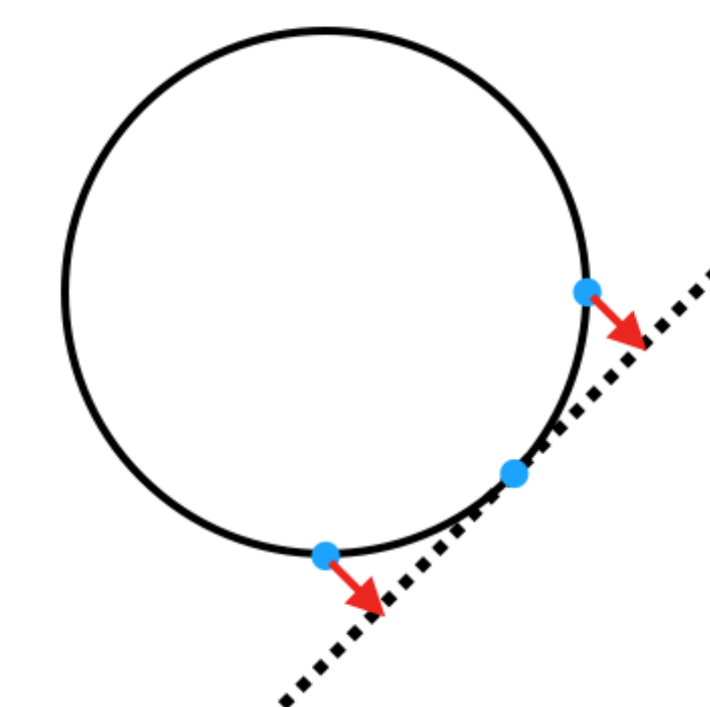
# Grid Points	L_∞ Error
400	0.5612
1600	0.4512
6400	0.0960
25600	0.0537

Horn Torus

$$\begin{array}{l} \text{Solve } -\Delta u + u = x \\ \text{on } (x^2 + y^2 + z^2)^2 - 4(x^2 + y^2) = 0 \end{array}$$

Smooth Regions

- Locally choose neighbors based on discretization and stencil
- Project all points for stencil onto tangent line or plane
- Finite difference method is used based on discretization and parametrization of curve/surface

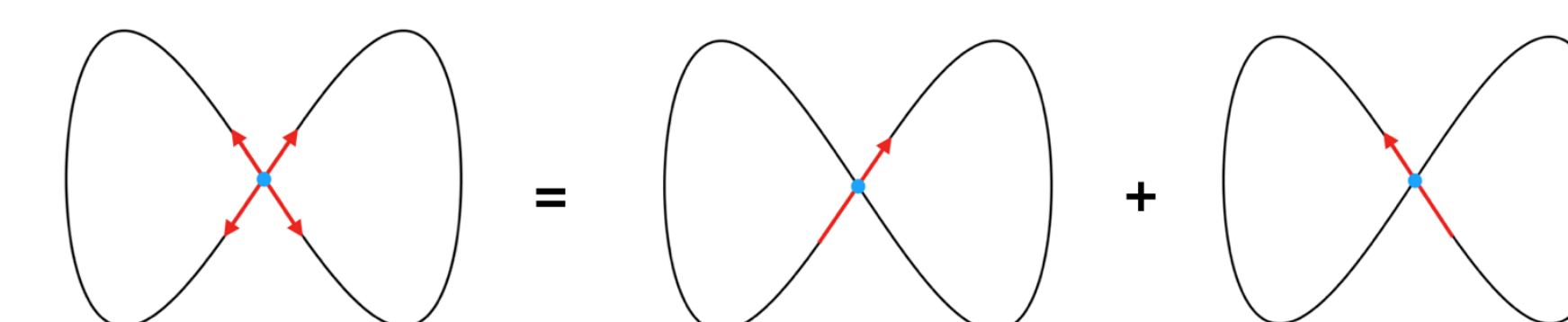


Theorem. Using a global parametrization, the numerical scheme is convergent and of second order.

Theorem. Using a local tangential parametrization, the numerical scheme is convergent and at least first order in arc-length mesh size.

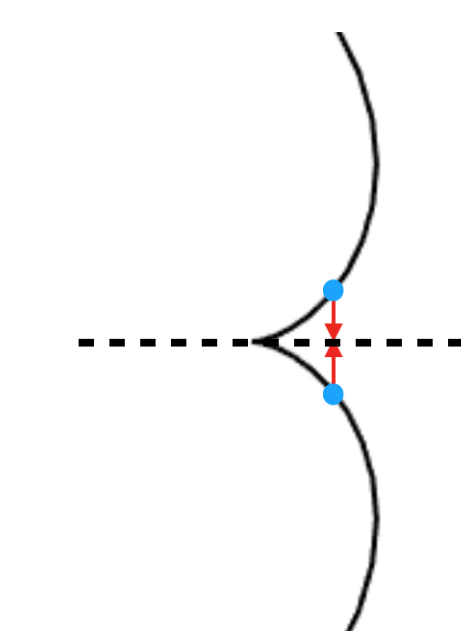
Singular Regions

For regions of the algebraic curve/surface with a singularity, local geometry needs to be analyzed for a more appropriate reparametrization. Treat each locally irreducible component as a different region to parametrize



Issues that arise with a cusp:

- No well-defined tangent direction at singular point
- Tangent-cone direction cannot be used



Locally reparametrize with a Puiseux series expansion.