

Goal

The complex monodromy group encodes structural information about the solutions to a polynomial system:

1. symmetry of solutions,
2. some restrictions to number of real solutions,
3. decomposition of varieties into irreducible components.

Benefits over $\mathbb{C}$ include:

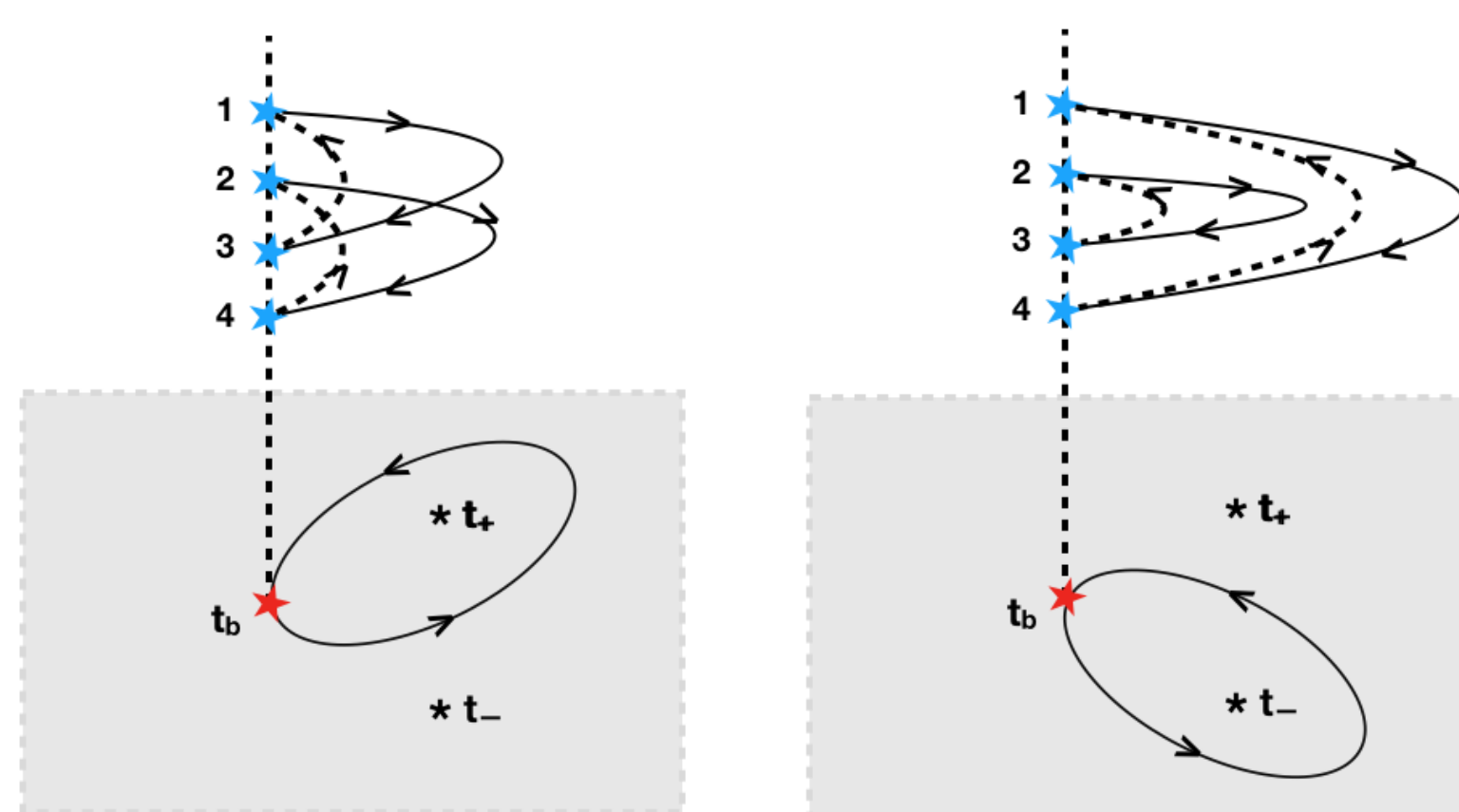
- method is independent of choice of basepoint,
- monodromy group is equivalent when a general curve section of the parameter space is considered.

Main goal: Extend monodromy action to $\mathbb{R}$ in 2 ways:

1. real monodromy group
2. real monodromy structure

This computation over $\mathbb{R}$ is achieved using numerical algebraic geometry within **Bertini**¹.

Complex Monodromy Group



Consider the parameterized polynomial system

$$F(x; p) = \begin{bmatrix} x_1^2 - x_2^2 - p_1 \\ 2x_1x_2 - p_2 \end{bmatrix} = 0.$$

- Fix basepoint $b = (1, 0) \in \mathbb{C}^2$ such that $p_1^2 + p_2^2 \neq 0$.
- Order the nonsingular isolated solutions:
 $x^{(1)} = (1, 0)$, $x^{(2)} = (-1, 0)$, $x^{(3)} = (0, i)$, $x^{(4)} = (0, -i)$
- Restrict the parameter space to the line parametrized by $\ell(t) = (1 - t, 2t)$ with $t_b = 0$.
- Loop around singular points to find permutations:
 $\sigma_{\gamma_+} = (1\ 3)(2\ 4)$ and $\sigma_{\gamma_-} = (1\ 4)(2\ 3)$

The complex monodromy group is the Klein group on four elements $K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2 \subset S_4$.

Real Monodromy Structure²

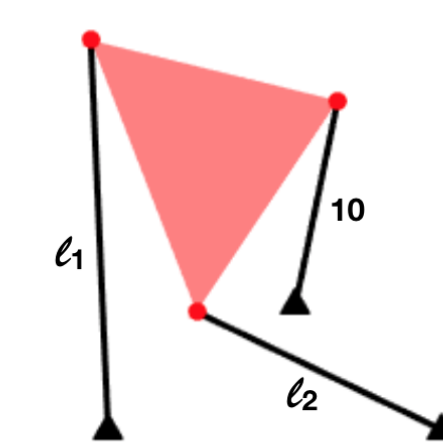
Consider the param. polynomial system

$$F(x; p) = \begin{bmatrix} (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\ 2x_1x_2 - p_2 \end{bmatrix} = 0.$$

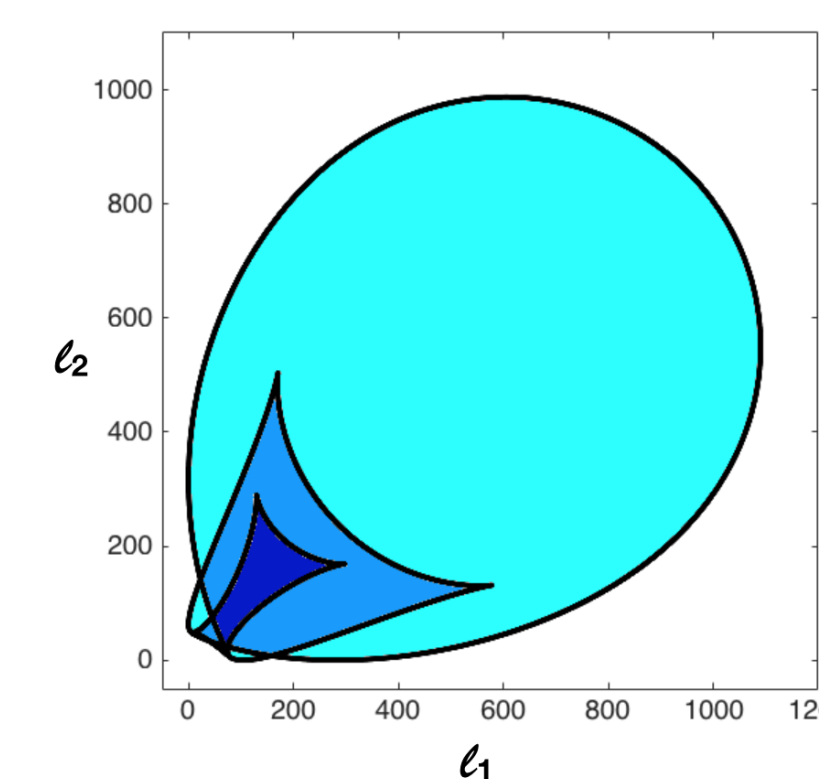
Real monodromy group is trivial, yet the *real monodromy structure* is not due to the ability to permute solutions 1 and 2, as seen in $\mathcal{G}_\bullet = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4\}$:

- | | |
|---|---|
| <ul style="list-style-type: none"> • $\mathcal{G}_1$ <ul style="list-style-type: none"> – $\{1\}, \{2\} \mapsto \{\{1\}, \{2\}\}$ – $\{q_1\} \mapsto \{\{q_1\}\}$ for all others • $\mathcal{G}_3$ <ul style="list-style-type: none"> – $\{q_1, q_2, q_3\} \mapsto \{\{q_1, q_2, q_3\}\}$ | <ul style="list-style-type: none"> • $\mathcal{G}_2$ <ul style="list-style-type: none"> – $\{1, 2\} \mapsto \{\{1, 2\}, \{2, 1\}\}$ – $\{q_1, q_2\} \mapsto \{\{q_1, q_2\}\}$ for all others • $\mathcal{G}_4$ <ul style="list-style-type: none"> – $\{q_1, q_2, q_3, q_4\} \mapsto \{\{q_1, q_2, q_3, q_4\}\}$ |
|---|---|

Application - 3-RPR³

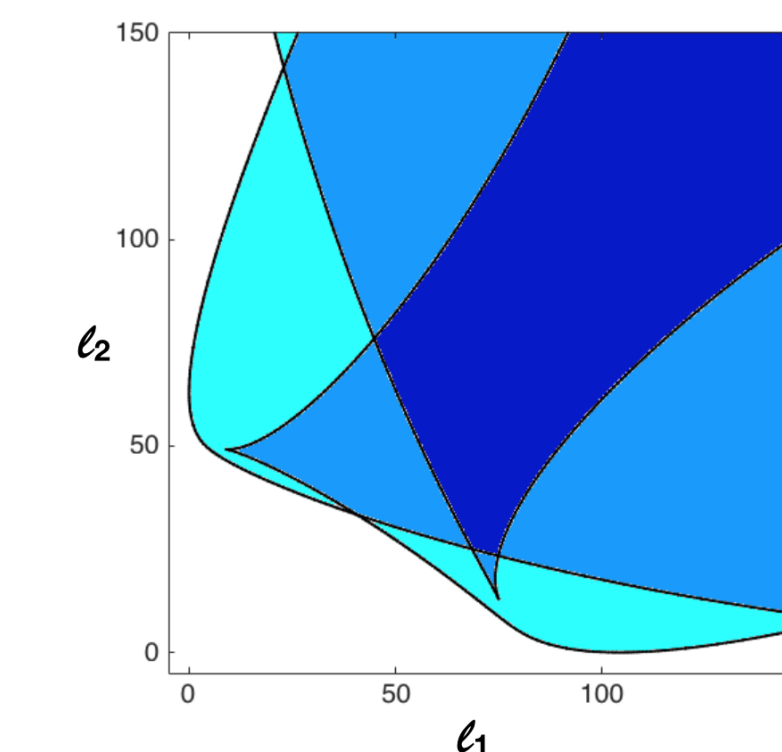


Sample 3-RPR mechanism with 1 leg length fixed.



Regions of param. space:

- navy = 6 real sols.
- grey blue = 4 real sols.
- baby blue = 2 real sols.
- white = 0 real sols.



Selected marked points (red star) in each connected component and all intermediary points (red dot).

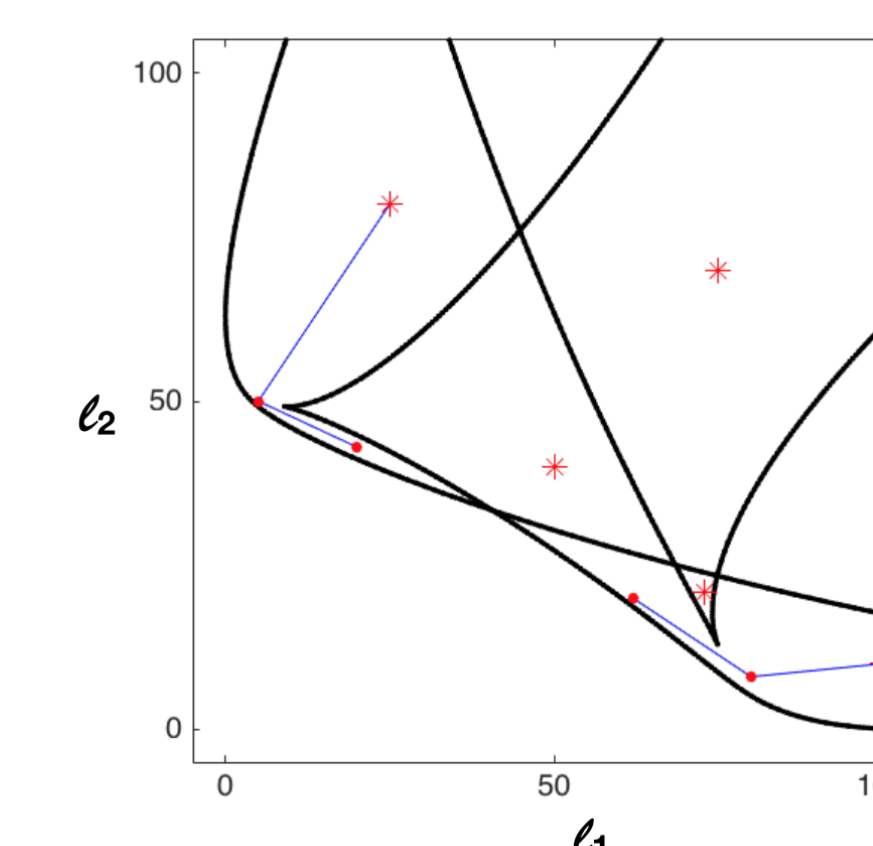
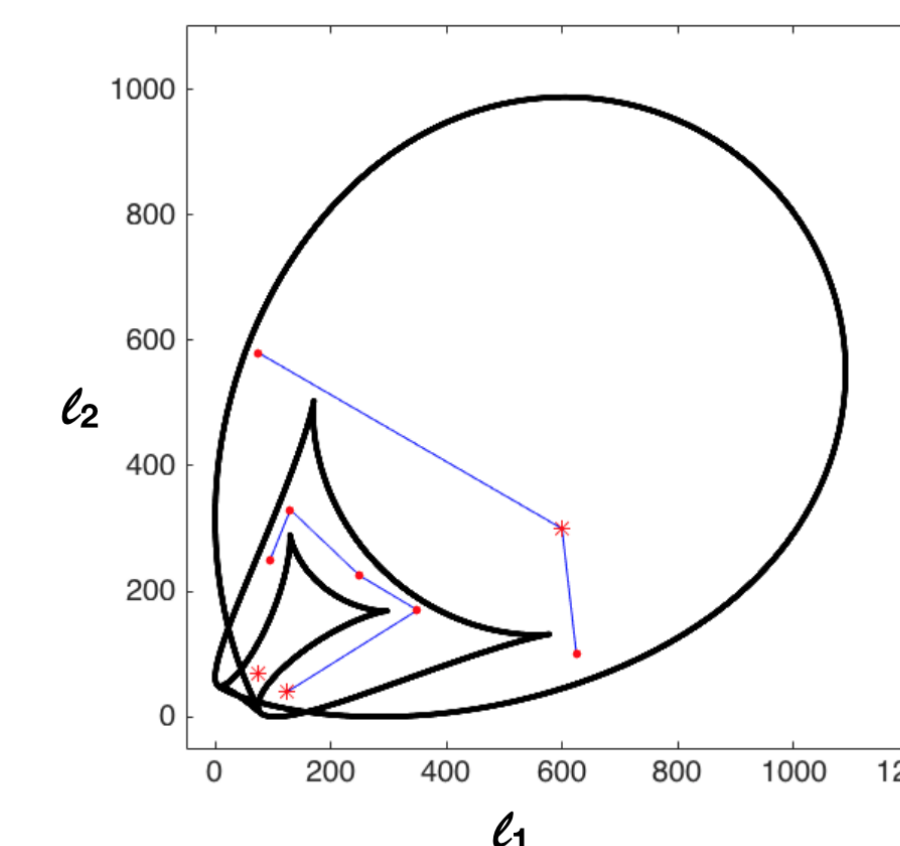
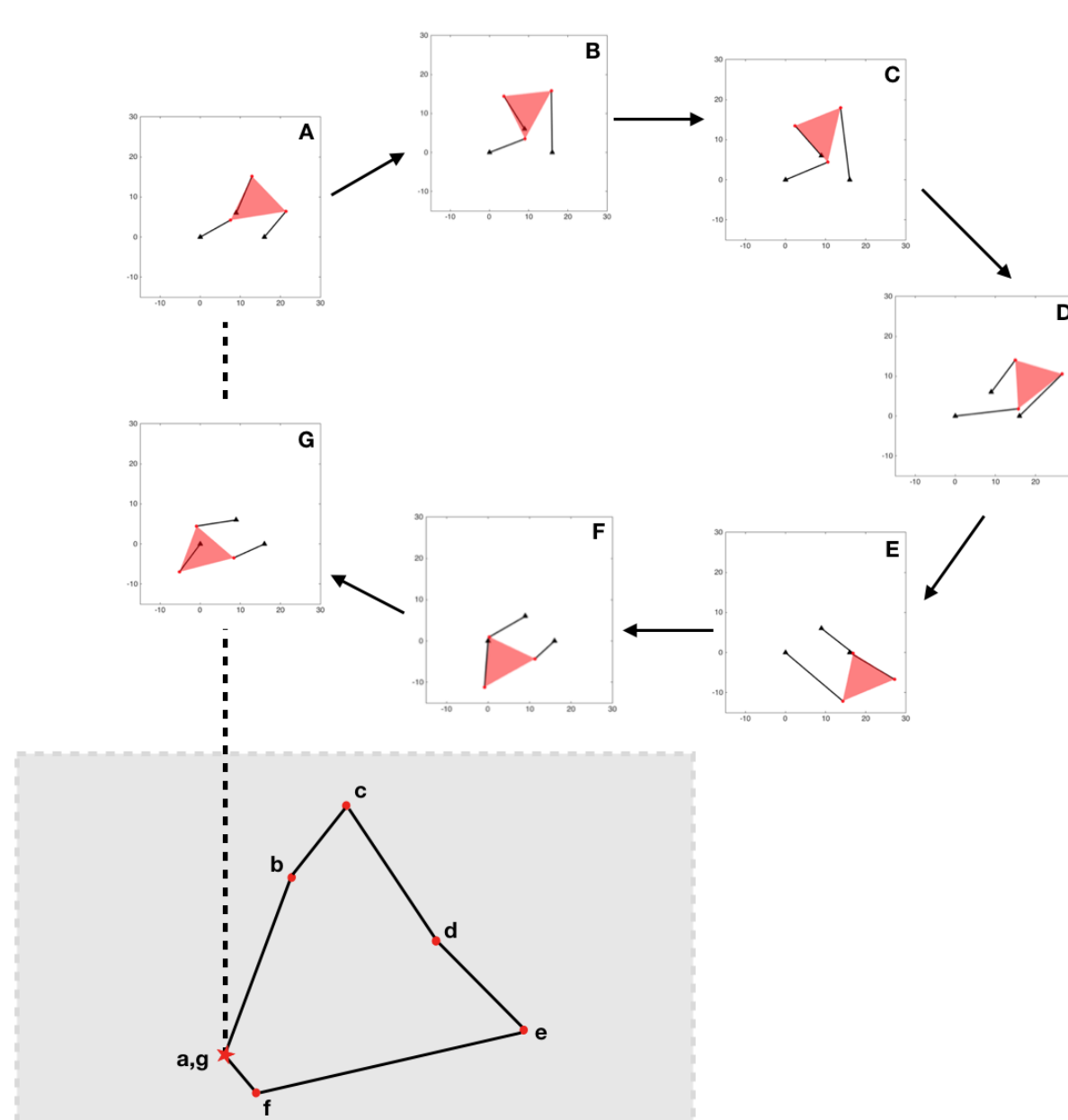


Illustration of a nonsingular assembly mode change between solutions 4 and 5 (left). This is encoded in $\mathcal{G}_1$ below:

- $\{1\}, \{2\}, \{3\} \mapsto \{\{1\}, \{2\}, \{3\}\}$
- $\{4\}, \{5\}, \{6\} \mapsto \{\{4\}, \{5\}, \{6\}\}$

Real Monodromy Group²

Consider the parameterized polynomial system

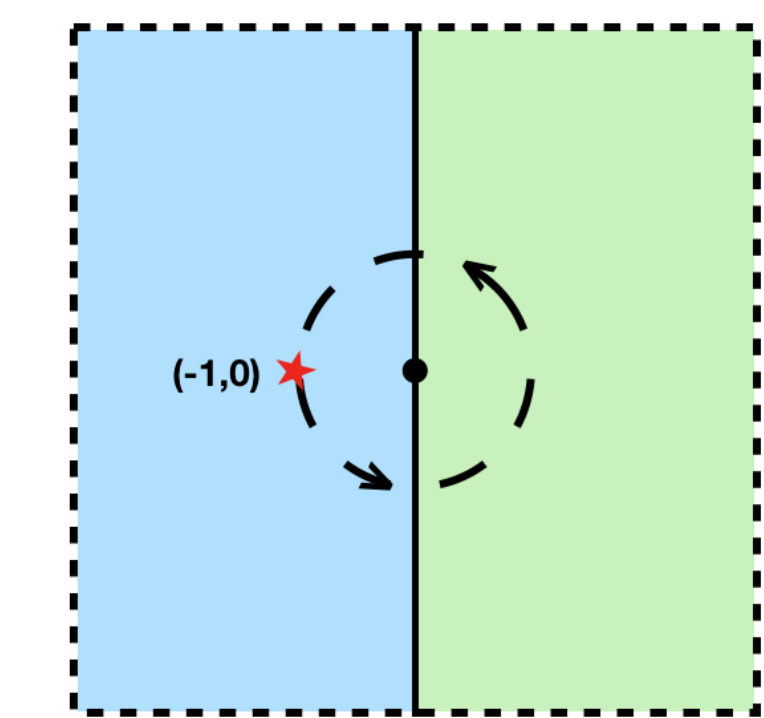
$$F(x; p) = \begin{bmatrix} x_1^2 - x_2^2 - p_1 \\ 2x_1x_2 - p_2 \end{bmatrix} = 0.$$

- Fix basepoint $b = (1, 0) \in \mathbb{R}^2$ such that $p_1^2 + p_2^2 \neq 0$.
- Order the **real** nonsingular isolated solutions:
 $x^{(1)} = (1, 0)$, $x^{(2)} = (-1, 0)$
- **Real** loop around singular point to find permutations:
 $\sigma_\gamma = (1\ 2)$

The real monodromy group is $\mathcal{S}_2 = \{(1), (1\ 2)\}$.

Consider a modified parameterized polynomial system

$$F(x; p) = \begin{bmatrix} (x_1^2 - x_2^2 - p_1)(x_1^2 + p_1) \\ 2x_1x_2 - p_2 \end{bmatrix} = 0.$$



- Fix basepoint $b = (-1, 0) \in \mathbb{R}^2$ such that $p_1^2 + p_2^2 \neq 0$.
- Order the **real** nonsingular isolated solutions:
 $x^{(1)} = (1, 0)$, $x^{(2)} = (-1, 0)$, $x^{(3)} = (0, 1)$, $x^{(4)} = (0, -1)$
- No nontrivial **real** loop exists for all 4 solutions.
- Fundamental group is trivial $\Rightarrow$ the real monodromy group is trivial

Shortcomings:

- dependent on choice of basepoint
- no possibility to slice parameter space
- restrictive - typically trivial

References

1. D. Bates, J. Hauenstein, A. Sommese, and C. Wampler. Bertini: Software for numerical algebraic geometry. Available at bertini.nd.edu.
2. J. Hauenstein and M. Regan. Real monodromy action. *Submitted*.
3. M.L. Husty (2009). Non-singular assembly mode change in 3-RPR-parallel manipulators. In *Computational Kinematics*, pp. 51–60.